MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2019

Calculator-free

Marking Key

The release date for this exam and marking scheme is 14th June.

Question 1(a) **(3 marks)**

Solution
If $$z = \sqrt{3} - i \quad \Rightarrow \quad z^2 = (\sqrt{3} - i)(\sqrt{3} - i) = 3 - 2\sqrt{3}i - 1 = 2(1 - \sqrt{3}i)$$ then $$z^3 = 2(1 - \sqrt{3}i)(\sqrt{3} - i) = 2(\sqrt{3} - 3i - i - \sqrt{3}) = -8i$$ Alternatively we note that $z = re^{i\theta}$ with $r = 2$ and $\theta = -\pi / 6$. Then $z^3 = r^3 e^{3i\theta} = 2^3 e^{-i\pi/2} = -8i$ as before

Mathematical behaviours	Marks
• calculates z^2 correctly	1
• calculates z^3 correctly (1 mark for showing real part zero and 1 mark for correct value of imaginary part)	1+1

Question 1(b) **(1 mark)**

Solution
Since $$z^3 = -8i$$ then $z^6 = -64$ which is real and negative. Hence $N = 6$

Mathematical behaviours	Marks
• calculates z^6 correctly	1

Question 2 (a) **(1 mark)**

Solution	
Augmented matrix = $\left[\begin{array}{ccc\|c} 3 & 3 & 3 & 3 \\ 6 & 10 & 10 & 9 \\ -3 & -4 & a & b \end{array}\right]$	
Mathematical behaviours	Marks
• correctly transfers coefficients of equations to augmented matrix	1

Question 2 (b) **(3 marks)**

Solution	
After $r_2 - 2r_1$ and $r_3 + r_1$ the system is reduced to: $\left[\begin{array}{ccc\|c} 3 & 3 & 3 & 3 \\ 0 & 4 & 4 & 3 \\ 0 & -1 & a+3 & b+3 \end{array}\right]$ After $r_2 + 4r_3$ the system is further reduced to $\left[\begin{array}{ccc\|c} 3 & 3 & 3 & 3 \\ 0 & 4 & 4 & 3 \\ 0 & 0 & 4a+16 & 4b+15 \end{array}\right]$	
Mathematical behaviours	Marks
• correctly reduces *x*-components to 0 for rows 2 and 3 (or equivalent) • correctly reduces *y*-component to 0 for row 3 (or equivalent)	2 1

Question 2 (c) **(3 marks)**

Solution	
From the augmented matrix form we deduce that (i) for unique solution, $a \neq -4$ (ii) for no solution $a = -4$ and $b \neq -\frac{15}{4}$ (iii) for infinitely many solutions $a = -4$ and $b = -\frac{15}{4}$	
Mathematical behaviours	Marks
• correctly determines value of a for a unique solution • correctly determines values of a and b that means there is no solution • correctly states the values of a and b for infinitely many solutions	1 1 1

Question 2 (d) **(3 marks)**

Solution
When $a=-4$ and $b=-15/4$ the augmented matrix becomes $$\left[\begin{array}{ccc\|c} 3 & 3 & 3 & 3 \\ 0 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Then the second equation gives $y=-z+\frac{3}{4}$ and first equation then leads to $3x=3-3z+3z-\frac{9}{4} \Rightarrow x=\frac{1}{4}$ Hence the general solution of the equations is $x=\frac{1}{4},\ y=\frac{3}{4}-\lambda,\ z=\lambda$

Mathematical behaviours	Marks
• determines y in terms of z (or vice-versa)	1
• determines the value of x	1
• states the general solution in terms of a suitable parameter	1

Question 3 **(5 marks)**

Solution
If $y=ax+b$ then $x=(y-b)/a$ so that $f^{-1}(x)=\frac{1}{a}x-\frac{b}{a} \quad (a \neq 0)$ For this to be the same as the linear function $f(x)=ax+b$ then comparison of the coefficients of x and the constant requires that $$a=\frac{1}{a} \quad \text{and} \quad b=-\frac{b}{a}$$ Hence $a^2=1$ so $a=\pm 1$. If $a=1$ then $b=-b$ so $b=0$ If $a=-1$ then b is arbitrary   We conclude that either $a=1,\ b=0$ or $a=-1$ with b any real number

Mathematical behaviours	Marks
• derives equation for the inverse	1
• compares coefficients to determine the equations for a and b	1
• solves for a	1
• derives correct solution for $a=1$	1
• dervies correct solution for $a=-1$	1

Question 4 **(4 marks)**

Solution
Let $z=\alpha+i\beta$ in which case $\bar{z}=\alpha-i\beta$; additionally $\lvert z\rvert^2=\alpha^2+\beta^2$ Now $$\frac{1}{z}=\frac{1}{\alpha+i\beta}=\frac{\alpha-i\beta}{(\alpha+i\beta)(\alpha-i\beta)}=\frac{\alpha-i\beta}{\alpha^2+\beta^2}=\frac{\bar{z}}{\lvert z\rvert^2}$$ as required

Mathematical behaviours	Marks
• writes down an appropriate form for z and hence $\bar{z}$	1
• derives an expression for $\lvert z\rvert^2$	1
• in quotient multiplies through by the complex conjugate	1
• draws a valid conclusion	1

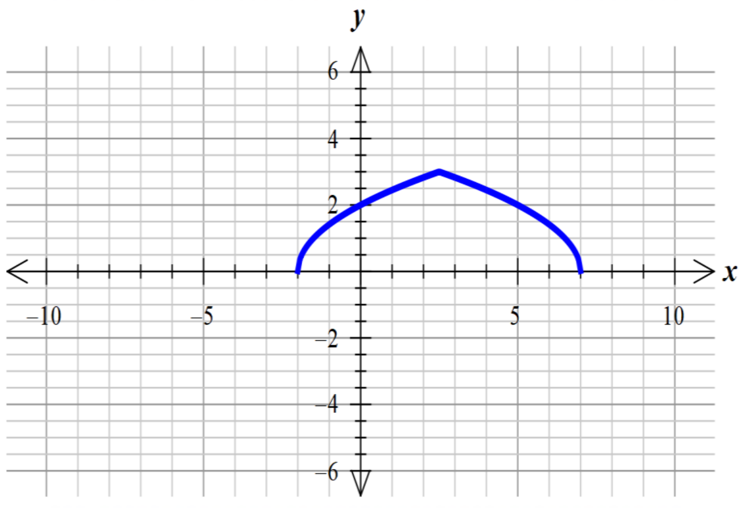
Question 5 (a) **(3 marks)**

Solution
$f(x)=\sqrt{9-\|5-2x\|}$ is defined if $9-\|5-2x\|\ge 0$ If $x\le 2.5$ then $9-\|5-2x\|=9-(5-2x)=2x+4\ge 0$ if $x\ge -2$ If $x\ge 2.5$ then $9-\|5-2x\|=9-(2x-5)=14-2x\ge 0$ if $x\le 7$ So $f(x)$ is defined for $-2\le x\le 7$

Mathematical behaviours	Marks
• obtains positivity requirement for $9-\|5-2x\|$	1
• obtains lower and upper limits of the domain	1+1

Question 5 (b) **(3 marks)**

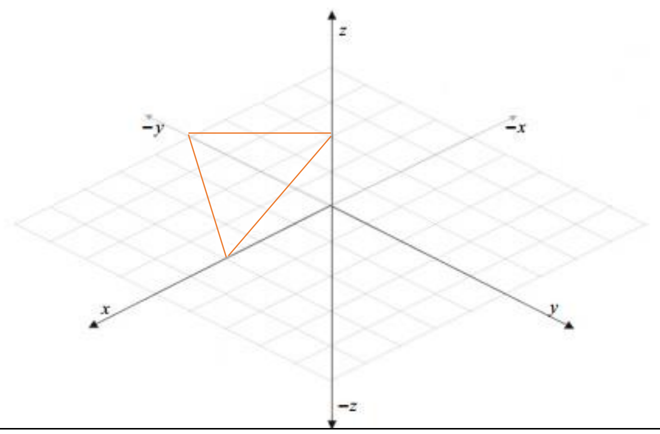
Solution



Mathematical behaviours	Marks
• displays general shape of the graph	1
• indicates maximum at $(2.5,3)$	1
• makes clear the non-differentiability at the maximum point	1

Question 6 (a) **(2 marks)**

Solution



Mathematical behaviours	Marks
• correctly sketches the triangle on the plane with all three intercepts cutting the axes at $x=3, y=-4, z=2$ (-1 for one mistake)	2

Question 6 (b) **(1 mark)**

Solution

Substituting gives: $4(3)-3(4)+6(2)=12$

Now LHS $=4(3)-3(4)+6(2)=RHS$ so the given point lies on the plane

Mathematical behaviours	Marks
• correctly substitutes point into equation and confirms value	1

Question 6 (c) **(2 marks)**

Solution

The vector **q** = $4\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ is perpendicular to Q.

Since **v** $=-2$ **q**, it can be concluded that **v** is parallel to **q**

As such **v** must also be perpendicular to Q.

Mathematical behaviours	Marks
• recognises that **v** is a scalar multiple of **w**	1
• concludes that **v** is parallel to **w** and so must also be perpendicular to Q	1

Question 6 (d) **(2 marks)**

Solution

Equation for $\pi_R: 4(x-4)-3(y-2)+6(z+3)=0$ so that $4x-3y+6z=-8$

Mathematical behaviours	Marks
• writes down equation with same coefficients $(4,-3,6)$	1
• shows how to incorporate the fact that the required plane includes the given point	1

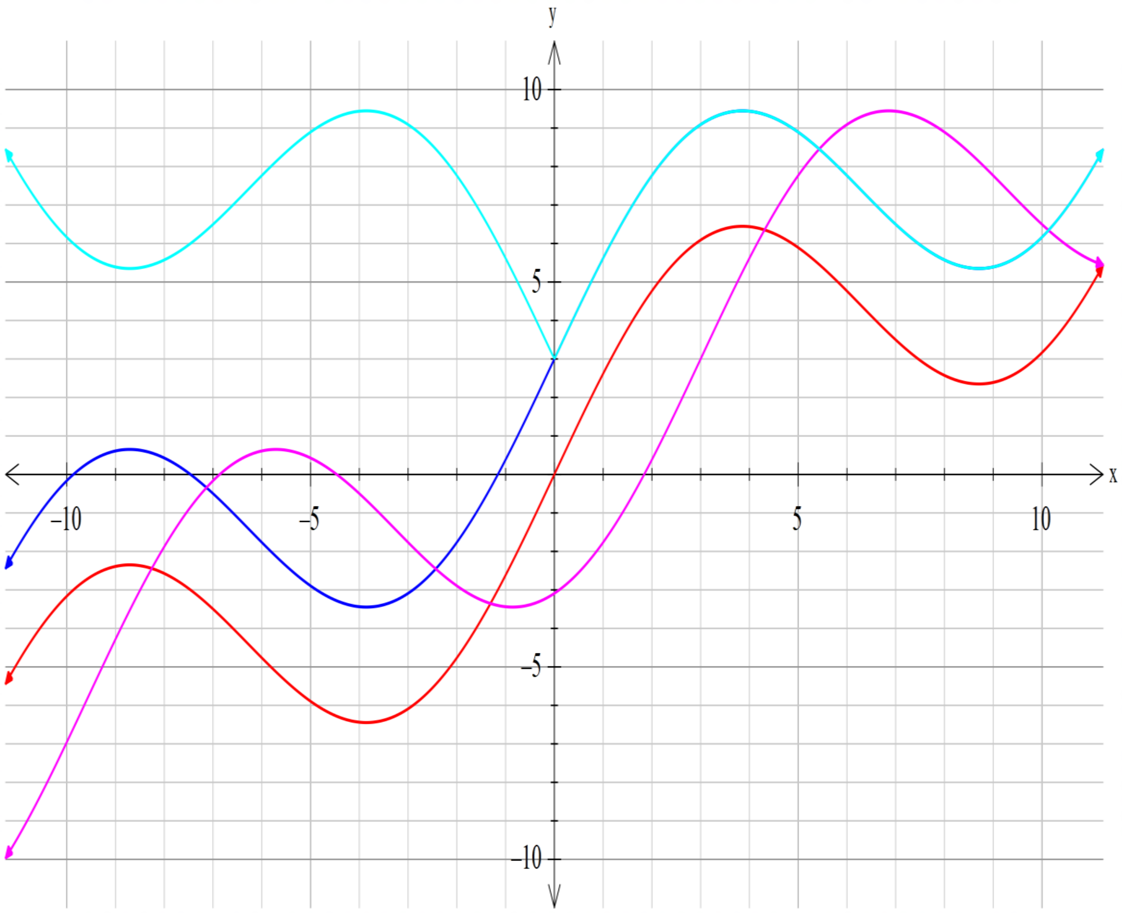
Question 6 (e) **(2 marks)**

Solution	
We can find **w** by forming the vector product $(3,4,2)\times(-8,6,-12)=(-60,20,50)$ This vector, or any non-zero multiple of it, is the required perpendicular vector.	
Mathematical behaviours	Marks
• makes clear the need to construct a vector product • computes the vector product correctly	1 1

Question 7 **(6 marks)**

Solution	
First note that $$4(1-i)=4\times\sqrt{2}\exp\left(-\frac{i\pi}{4}\right)=2^{5/2}\exp\left(-\frac{i\pi}{4}\right)$$ Then $z^5=2^{5/2}\exp\left(i\pi\left[2k-\frac{1}{4}\right]\right)\Rightarrow z=\sqrt{2}\exp\left(\frac{i\pi}{5}\left[2k-\frac{1}{4}\right]\right)$ for $k=0...4$ by de Moivre's theorem Hence the five roots are $z=\sqrt{2}\exp(i\vartheta)$ where $\vartheta=-\frac{\pi}{20},\frac{7\pi}{20},\frac{3\pi}{4},\frac{23\pi}{20},\frac{31\pi}{20}.$ Restricting the argument to the stated domain leaves $\vartheta=-\frac{17\pi}{20},-\frac{9\pi}{20},-\frac{\pi}{20},\frac{7\pi}{20},\frac{3\pi}{4}.$	
Mathematical behaviours	Marks
• writes $4(1-i)$ in a suitable polar form (1 for modulus, 1 for argument) • uses de Moivre's theorem appropriately • writes down the five required roots (-1 for one mistake) • calculates all the arguments so that they lie in the appropriate given range	1+1 1 2 1

Question 8 **(6 marks)**

Solution

The graph of $f(x-3)$ is obtained by shifting the graph of $f(x)$ 3 units to the right. The graph of $f(x)-3$ is obtained by shifting the graph of $f(x)$ 3 units down. The graph of $f(\lvert x\rvert)$ is the same as the graph of $f(x)$ for $x \geq 0$ and then that part is reflected across the y-axis.

Mathematical behaviours	Marks
• displays the correct geometric transformations	1+1+1
• plots the graphs reasonably accurately	1+1+1